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ON THE MOTION OF KOWALEWSKA'S GYROSCOPE IN THE DELONE CASE

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The general qualitative portrait of the motion of a heavy rigid body around a fixed point was ascertained in [1 - 4] under Kowalewska's assumption in the Delone case. In this paper the motion of a rigid body is investigated under the assumption that the body is imparted a large angular velocity around an axis close to the major axis of the energy ellipsoid. The explicit dependencies of the Euler angles on time, obtained here, permit us to carry out sufficiently simply a similar analysis of the motion of Kowalewska's gyroscope in the Delone case.

1. As is known [1, 2, 5], the equations of motion of a heavy rigid body around a fixed point, under Kowalewska's assumption

$$A = B = 2C, \qquad y_0 = z_0 = 0, \qquad c = Mgx_0C^{-1} \neq 0$$
$$2p^{\cdot} = qr, \qquad 2q^{\cdot} = -pr - c\gamma'', \qquad r^{\cdot} = c\gamma'$$
$$\gamma^{\cdot} = r\gamma' - q\gamma'', \qquad \gamma'^{\cdot} = p\gamma'' - r\gamma, \qquad \gamma''^{\cdot} = q\gamma - p\gamma' \tag{1.1}$$

possess, under definite conditions, as was noted by Delone, the five algebraic integrals

$$2p^2 + 2q^2 + r^2 = 2c\gamma - 6l', \qquad 2p\gamma + 2q\gamma' + r\gamma'' = 2l \tag{1.2}$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1, \qquad p^2 - q^2 + c\gamma = 0, \qquad 2pq + c\gamma' = 0$$

(l', l are arbitrary constants), and the general solution of these equations (1.1) can be expressed in elliptic time functions.

Let us assume that at the initial instant the body's principal inertial axis Oy lies in the horizontal plane, the principal inertial axis Oz makes an angle θ_0, $0 < \theta_0 \leqslant \pi / 2$ with the vertical (the case $\theta_0 = 0$ will be treated below), and the projection of the angular velocity onto the axis Oz is a large quantity. Then

$$\gamma_0 = \sin\,\theta_0, \qquad \gamma_0' = 0, \qquad \gamma_0'' = \cos\theta_0 \tag{1.3}$$

while the last two relations in (1.2) are satisfied under the conditions

$$p_0 = 0, \qquad q_0^2 = c\gamma_0 \tag{1.4}$$

Assume that

$$c > 0, \qquad r_0 > 0, \qquad q_0 > 0 \tag{1.5}$$

Here and below we have introduced the notation $F_0 = F(0)$ for any function $F(t)$. By using from relations (1.2) the formulas [2]

$$4p^2 + r^2 = 6l', \qquad \gamma'' = 2r^{-1}[l + c^{-1}p(p^2 + q^2)] \tag{1.6}$$

$$p^2 + q^2 = -\frac{2lc}{3l'}\,p + \frac{c}{6l'}\,r\,\sqrt{6l' - 4l^2} \tag{1.7}$$

in which, by virtue of conditions (1.3) and (1.4), we need to set

$$6l' = r_0^2, \qquad 2l = r_0\gamma_0'', \qquad 6l' - 4l^2 = r_0^2\,\gamma_0^2 \tag{1.8}$$

we have, on the basis of (1.5) and (1.7),

$$q = (-p^2 - 2c\gamma_0'' r_0^{-1}\, p + c\gamma_0 r_0^{-1} r)^{1/2} \tag{1.9}$$

By introducing the new variable σ and the parameter μ

$$\sqrt{r_0^2 - 4p^2} = r_0 + 2p\mu^{-1}\sigma, \qquad \mu = \sqrt{c\gamma_0}\,/\,r_0 \quad (\sigma_0 = 0) \tag{1.10}$$

from relations (1.6), (1.9), (1.10) and the first equation of system (1.1), we obtain

$$p = -r_0\mu\,(\mu^2 + \sigma^2)^{-1}\sigma, \qquad q = r_0\mu\,(\mu^2 + \sigma^2)^{-1}\sqrt{R(\sigma)}$$

$$r = r_0\,(\mu^2 - \sigma^2)(\mu^2 + \sigma^2)^{-1} \tag{1.11}$$

$$2\sigma^{\cdot} = -r_0\sqrt{R(\sigma)}, \qquad R(\sigma) = -(\sigma^4 - 2\alpha\mu\sigma^3 + \sigma^2 - 2\alpha\mu^3\sigma - \mu^4)$$

$$\alpha = \gamma_0''\gamma_0^{-1} \tag{1.12}$$

We determine the dependency of the variable σ on time by Sretenskii's method [6]. For this purpose we find the expansion of the roots σ_1, σ_2, σ_3, σ_4 of the equation $R(\sigma) = 0$ into series in the small parameter μ :

$$\sigma_1 = -\mu^2 + \alpha\mu^3 + O(\mu^4), \qquad \sigma_2 = \mu^2 + \alpha\mu^3 + O(\mu^4) \tag{1.13}$$

$$\sigma_3 = -i + \mu\alpha + O(\mu^2), \qquad \sigma_4 = i + \mu\alpha + O(\mu^2)$$

and we pass to the new variable ν

$$2\sigma = (\sigma_1 + \sigma_2) + (\sigma_2 - \sigma_1)\cos\nu = \mu^2\cos\nu + \mu^3\alpha + O(\mu^4) \tag{1.14}$$

$$\nu_0 = {}^1\!/_2\pi + \mu\alpha + O(\mu^2)$$

Substituting relations (1.13), (1.14) into Eq. (1.12) and integrating it, we have

$$\nu = {}^1\!/_2\pi + {}^1\!/_2 r_0 t + O(\mu) \tag{1.15}$$

From formulas (1.2), (1.6), (1.11), (1.14) we obtain

$$p = -\sqrt{c\gamma_0}\,(\cos\nu + \mu\alpha) + O(\mu^2), \qquad q = \sqrt{c\gamma_0}\sin\nu + O(\mu^2)$$

$$r = r_0 + O(\mu)$$

$$\gamma = -\gamma_0(\cos 2\nu + 2\mu\alpha\cos\nu), \qquad \gamma' = \gamma_0(\sin 2\nu + 2\mu\alpha\sin\nu)$$

$$\gamma'' = \gamma_0'' - 2\mu\gamma_0\cos\nu + O(\mu^2) \tag{1.16}$$

2. For the analysis of the motion we introduce the Euler angles θ, φ, ψ

$$\cos\theta = \gamma'', \qquad \psi^{\cdot} = (p\gamma + q\gamma')(1 - \gamma''^2)^{-1}, \qquad \varphi^{\cdot} = r - \psi^{\cdot}\cos\theta \tag{2.1}$$

From the first formula in (2.1) and the last formula in (1.16) follows the expression for the nutation angle:

$$\theta = \theta_0 + 2\mu\cos\nu + O(\mu^2) \tag{2.2}$$

By substituting expressions (1.16) into the second formula of (2.1) which we rewrite, on the basis of relations (1.2), in the form $\psi^{\cdot} = -cp(p^2 + q^2)^{-1}$, and integrating, we obtain a formula for the precession angle

$$\psi = \psi_0 + 2\mu\gamma_0^{-1}\sin\nu + \mu^2\alpha\gamma_0^{-1}\sin 2\nu + O(\mu^3) \tag{2.3}$$

We find the angle of natural rotation from the third relation in (2.1)

$$\varphi = {}^1\!/_2\pi + r_0 t + O(\mu) \tag{2.4}$$

To determine the motion of Kowalewska's gyroscope in the Delone case with the aid of formulas (2.2) - (2.4) we take, on a fixed unit sphere with center at a fixed point, a spherical rectangle formed by parallels distant from the mean parallel θ_0 by angles $\pm 2\mu$ and by meridians distant from the mean meridian $(\psi_0 - {}^1/_2\pi)$ by angles $\pm 2\mu\gamma_0^{-1}$. Then the trajectory of the axis Oz on the unit sphere indicated is the ellipse

$$\frac{\theta_1^2}{4\mu^2} + \frac{\psi_1^2}{4\mu^2\gamma_0^{-2}} = 1 \qquad (\theta_1 = \theta - \theta_0, \psi_1 = \psi - \psi_0) \tag{2.5}$$

In describing this ellipse the gyroscope's axis Oz executes, in the first approximation, a periodic motion with period $T = 4\pi / r_0$, passing at the instants t_n and t_m

$$t_n = 2\pi n r_0^{-1}, \qquad t_m = (2m + 1)\pi r_0^{-1} \, (m,n = 0, \pm 1,\ldots)$$

through the points of intersection of the mean parallel with the extreme meridians and of the mean meridian with the extreme parallels. As follows from formula (2.4) the natural rotation of the body differs but little from the uniform rotation with large angular velocity r_0.

3. Let us now consider the motion of the gyroscope for the condition $\theta_0 = 0$ from which, with due regard to formulas (1.3), (1.4) follow the relations

$$p_0 = q_0 = \gamma_0 = \gamma_0' = 0, \qquad \gamma_0'' = 1, \qquad 6l' - 4l^2 = 0 \tag{3.1}$$

Then, from formulas (1.5) - (1.7) we have

$$q = -\sqrt{-p(p + 2\mu_1\sqrt{c})}, \quad r = r_0\sqrt{1 - 4\mu_1^2c^{-1}p^2}, \quad \mu_1 = \sqrt{c}/r_0 \tag{3.2}$$

The minus sign before the first radical was chosen by virtue of the condition $q_0 = 0$ and of the condition $q_0' = -\gamma_0''c < 0$ obtained from the second equation of system (1.1) and relations (3.1) and (1.5).

To determine the dependency of p on time we rewrite the first of the equations of system (1.1), using relations (3.2), in the form

$$[f(\xi)]^{-1/2}d\xi = -{}^1/_2 r_0 dt, \qquad \xi = 2\mu_1 c^{-1/2}p$$
$$f(\xi) = (1 - \xi^2)(-\xi^2 - 4\mu_1^2\xi) \qquad (\xi_0 = 0) \tag{3.3}$$

and we pass, analogously to what we did above, to a new variable u

$$\xi = 2\mu_1^2(\cos u - 1) \qquad (u_0 = 0) \tag{3.4}$$

Substituting relation (3.4) into Eq. (3.3) and integrating it, we obtain

$$u = {}^1/_2 r_0 t + O(\mu_1^3)$$

From formulas (3.2) - (3.4) and (1.2) we have

$$p = -\mu_1\sqrt{c}(1 - \cos u), \quad q = -\mu_1\sqrt{c}\sin u, \quad r = r_0 + O(\mu_1^3)$$
$$\gamma = -\mu_1^2(1 - 2\cos u + \cos 2u), \qquad \gamma' = -\mu_1^2(2\sin u - \sin 2u)$$
$$\gamma'' = 1 - 2\mu_1^4(1 - \cos u)^2 + O(\mu_1^8) \tag{3.5}$$

From the relations for the Euler angles θ, φ, ψ,

$$\theta = 2\mu_1^2(1 - \cos u) + O(\mu_1^6), \qquad \psi - \psi_0 = {}^1/_2 r_0 t$$
$$\varphi = {}^1/_2\pi + {}^1/_2 r_0 t + O(\mu_1^3) \tag{3.6}$$

obtained on the basis of formulas (2.1) and (3.5), it follows that the trajectory of the axis Oz on a fixed sphere of unit radius is a cardioid [7]

$$\theta = 2\lambda(1 - \cos\psi_1) \qquad (\lambda = \mu_1^2,\ \psi_1 = \psi - \psi_0).$$

In describing this cardioid the axis Oz executes, in the first approximation, a periodic motion of period $T = 4\pi / r_0$. The natural rotation of the body, as follows from the last formula in (3.6), differs but little from the uniform rotation with large angular velocity $^1/_2\, r_0$.

The analysis presented allows us to observe, to a sufficient degree, the motion of Kowalewska's gyroscope in the Delone case and to ascertain the dependency of this motion on the design parameters of the gyroscope and on the initial conditions of the motion.

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